

Regularity of generalized Daubechies wavelets reproducing exponential polynomials

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Abstract

We investigate non-stationary orthogonal wavelets based on a non-stationary interpolatory subdivision scheme reproducing a given set of exponentials. The construction is analogous to the construction of Daubechies wavelets using the subdivision scheme of Deslauriers-Dubuc. The main result is the smoothness of these Daubechies type wavelets.

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1 Introduction

The main purpose of the present paper is to study a non-stationary interpolatory subdivision scheme reproducing exponentials and the related Daubechies type wavelets.

In [9] G. Deslauriers and S. Dubuc investigated a subdivision scheme based on polynomial interpolation of odd degree $2n - 1$. They proved the existence of the basic limit function $\Phi^{D_{2n}}$ for the subdivision scheme and determined its order of regularity. Daubechies wavelets are closely related to this construction since the autocorrelation function of the Daubechies scaling function is equal to the basic limit function $\Phi^{D_{2n}}$ (see [6], [33], or Section 5 below). Moreover, the regularity of the Daubechies wavelets can be derived from the regularity of the basic limit function $\Phi^{D_{2n}}$.

The motivation for the present work originated from the attempt to define a new concept of multivariate subdivision scheme *in the spirit of Deslauriers and Dubuc*, which is based respectively on multivariate interpolation. According to the Polyharmonic Paradigm introduced in [19] polyharmonic interpolation on parallel hyperplanes (or on concentric spheres) provides a proper generalization of the one-dimensional polynomial interpolation and, respectively, generates a natural multivariate subdivision scheme. We shall present

in a forthcoming paper [13] the *polyharmonic subdivision scheme* as an immediate generalization of the one-dimensional subdivision scheme of Dubuc-Deslauriers. This subdivision scheme is **stationary** and reproduces polyharmonic functions of fixed order. It is reduced to an infinite family of one-dimensional inherently **non-stationary** subdivision schemes reproducing exponential polynomials of special type. Thus the results about subdivision schemes for exponentials are the building bricks of the polyharmonic subdivision schemes discussed in detail in [13], [20], [21].

On the other hand, the study of non-stationary subdivision schemes reproducing general exponential polynomials were initiated in 2003 in [14], where the main results of Deslauriers and Dubuc were generalized, in particular, existence of basic limit function and its regularity were proved. For a given set of numbers $\{\lambda_j\}_{j=0}^n$, called sometimes frequencies, such a scheme reproduces the space $V = \text{span} \{e^{\lambda_j t}\}_{j=0}^n$, and is characterized by a family of symbols $\{a^{[k]}(z)\}_{k \in \mathbb{Z}}$, where $a^{[k]}(z)$ defines the refinement rule at level k . Ch. Micchelli proved in [30] that the symbols of these schemes are non-negative on the unit circle, whenever the set of frequencies $\{\lambda_j\}_{j=0}^n$ are real and symmetric. Based on this property of the symbols, he applied in [30] the construction of Daubechies to the fixed symbol $a^{[0]}(z)$, corresponding to a set of exponentials defined by the frequencies $\{\lambda_1, \lambda_2, \dots, \lambda_n, -\lambda_1, -\lambda_2, \dots, -\lambda_n, 0\}$. However, the so constructed wavelets reproduce only the function $e^0 = 1$. In contrast to that, in the present paper we elaborate on a genuine non-stationary scheme: we apply the same construction to the family of symbols $\{a^{[k]}(z)\}_{k \in \mathbb{Z}}$ of a *non-stationary* scheme reproducing V , and show that the resulting non-stationary wavelets generate a *Multiresolution Analysis* (MRA) which contains V . Let us mention that in [36] a similar construction is proposed without the validation of the positivity of the symbols for all k , which is necessary for the construction; see more discussion on the above references at the end of the paper.

The main result of the present paper states that the order of regularity of the new Daubechies type wavelets reproducing $e^{\lambda_j x}$ for $j = 1, \dots, n$, is at least as large as in the case of classical Daubechies wavelets reproducing polynomials of degree $\leq n - 1$ *provided that* the filters of the Daubechies scheme are chosen at each level in an appropriate way. The proof of this result depends on the concept of asymptotically equivalent subdivision schemes developed in [16]. These regularity results are important for the regularity of the multivariate polyharmonic subdivision schemes and the corresponding multivariate wavelets considered in [13].

What concerns the computational aspects of the present results, let us mention that in the case of a general set of frequencies $\{\lambda_j\}_{j=1}^n$ it is difficult to find concise expressions for the symbols of subdivision and respectively for the filters of the wavelets, in particular for the analogs of the important classical polynomials Q_n (appearing in equality (25) below), and the polynomials $b^{[k]}(z)$ (in equality (32) below). However we have to note the remarkable fact that in the special cases of interest for the multivariate polyharmonic subdivision scheme the corresponding polynomials can be explicitly constructed, see [13], [20], [21], [22], [23]. Also, in [20], [21], [22], [23] the polyharmonic subdivision wavelets

of Daubechies type were successfully applied to problems of Image Processing, where the experiments were carried out on benchmark images as well as on some astronomical images.

Finally, let us mention that after [8] non-stationary Wavelet Analysis reproducing exponentials have appeared in a natural way in the monograph [19] in the context of multivariate polyspline wavelets.

The paper is organized as follows: in Section 2 we shall review basic notions for non-stationary subdivision schemes. Section 3 is devoted to subdivision schemes for exponential polynomials and we shall give an improved estimate of the order of regularity of the basic limit function. In Section 4 we shall briefly review non-stationary multiresolutional analysis. In Section 5 we shall give present the construction of the Daubechies scaling function via subdivision schemes and estimate the order of regularity of the new Daubechies type wavelets.

Finally, we introduce some notations: \mathbb{N}_0 denotes the set of all natural numbers including zero, \mathbb{Z} denotes the set of all integers and we define the grid $2^{-k}\mathbb{Z} = \{j/2^k : j \in \mathbb{Z}\}$. By $C^\ell(\mathbb{R})$ we denote the set of all functions $f : \mathbb{R} \rightarrow \mathbb{C}$ which are ℓ -times continuously differentiable where $\ell \in \mathbb{N}_0$. The Fourier transform of an integrable function $f : \mathbb{R} \rightarrow \mathbb{C}$ is defined by

$$\widehat{f}(\omega) = \int_{-\infty}^{\infty} f(x) e^{-ix\omega} dx.$$

2 Basics in nonstationary subdivision schemes

In this section we shall briefly recall notations and definitions used in non-stationary subdivision schemes for functions on the real line. The formal definition of a subdivision scheme is the following:

Definition 1 A (non-stationary) **subdivision scheme** S_0 is given by a family of sequences $(a_j^{[k]})_{j \in \mathbb{Z}}$ of complex numbers indexed by $k \in \mathbb{N}_0$, called the masks or rules at level k , such that $a_j^{[k]} \neq 0$ only for finitely many $j \in \mathbb{Z}$. Given a sequence of numbers $f^0(j), j \in \mathbb{Z}$, one defines inductively a sequence of functions $f^{k+1} : 2^{-(k+1)}\mathbb{Z} \rightarrow \mathbb{C}$ by the rule

$$f_j^{k+1} := f^{k+1}\left(\frac{j}{2^{k+1}}\right) = \sum_{l \in \mathbb{Z}} a_{j-2l}^{[k]} f^k\left(\frac{l}{2^k}\right) \quad \text{for } j \in \mathbb{Z}. \quad (1)$$

If for each $k \in \mathbb{N}_0$ the masks $a_j^{[k]}, j \in \mathbb{Z}$, are identical the scheme is said to be **stationary**. The subdivision scheme is called **interpolatory** if for all $k \geq 0$ and $j \in \mathbb{Z}$ holds

$$f_{2j}^{k+1} = f_j^k. \quad (2)$$

An important tool in subdivision schemes is the *symbol* $a^{[k]}$ of the rule, or mask $a_j^{[k]}, j \in \mathbb{Z}$, defined by

$$a^{[k]}(z) := \sum_{j \in \mathbb{Z}} a_j^{[k]} z^j$$

which is Laurent polynomial since we assume that the sequence $a_j^{[k]}, j \in \mathbb{Z}$, has finite support. We shall identify the subdivision scheme S_0 by its masks $a_j^{[k]}, j \in \mathbb{Z}, k \in \mathbb{N}_0$ or its symbols $a^{[k]}, k \in \mathbb{N}_0$. It is clear that a subdivision scheme is stationary if and only if $a^{[k]}(z) = a(z)$, for all $k \in \mathbb{N}_0$. Moreover the scheme is interpolatory if and only if $a_{2j}^{[k]} = \delta_{0,j}$, for all $j \in \mathbb{Z}$. In terms of the symbol $a^{[k]}$ it is easy to prove that this is equivalent to the identity

$$a^{[k]}(z) + a^{[k]}(-z) = 2, \quad z \in \mathbb{C} \setminus \{0\}.$$

Definition 2 Let $\ell \in \mathbb{N}_0$. A subdivision scheme S_0 is called C^ℓ -**convergent** if for any bounded initial sequence $\{f_j^0\}_{j \in \mathbb{Z}}$ there exists $F \in C^\ell(\mathbb{R})$ such that

$$\lim_{k \rightarrow \infty} \sup_{j \in \mathbb{Z}} \{|F(j2^{-k}) - f_j^k|\} = 0. \quad (3)$$

The function F is called **limit function of the subdivision scheme** for the initial sequence $\{f_j^0\}_{j \in \mathbb{Z}}$. The limit function for the initial data function $f_j^0 = \delta_{0j}$ (here δ_{0j} is the Kronecker symbol) is called **basic limit function** of the scheme and it is denoted by Φ_0 .

A subdivision scheme is called **convergent** if it is just C^0 -convergent. A central problem in the theory of subdivision schemes is to estimate the order of regularity of a limit function whenever it exists. Let us remark that for an *interpolatory* scheme the limit function F in Definition 2 is easily computed for dyadic numbers $t = \frac{j}{2^k}$ with $k \geq 0$ and $j \in \mathbb{Z}$ through the values f_j^k and convergence of the scheme asks whether it has a continuous extension over the whole \mathbb{R} .

Remark 3 It is easy to see that for a convergent subdivision scheme S_0 with symbols $a^{[k]}(z) := \sum_{j=-N}^N a_j^{[k]} z^j$ for all $k \in \mathbb{N}_0$, the support of the basic limit function Φ_0 is contained in $[-N, N]$.

In the non-stationary case the following concept is of importance:

Definition 4 Let S_0 be a subdivision scheme given by the masks $(a_j^{[k]})_{j \in \mathbb{Z}}$. For any natural number $m \in \mathbb{N}_0$ define a new subdivision scheme S_m by means of the masks of level k :

$$a^{[k],m}(z) := a^{[k+m]}(z) \text{ for all } k \in \mathbb{N}_0.$$

The following result is proved in [15]:

Theorem 5 *If S_0 is a convergent subdivision scheme then S_m is convergent for any $m \in \mathbb{N}_0$.*

*The basic limit function of this scheme is **denoted** by Φ_m .*

The following result is well-known, see e.g. the proof of Theorem 2.1 in [5], for further results see also [31].

Proposition 6 *Let S_0 be a subdivision scheme with symbols $a^{[k]}(z)$ for $k \in \mathbb{N}_0$ such that*
(i) $a_j^{[k]} = 0$ for all $|j| \geq N$ and all $k \in \mathbb{N}_0$ for some fixed integer $N > 0$.
(ii) there is a constant $M > 0$ such that $|a^{[k]}(e^{i\omega})| \leq M$ for all $k \in \mathbb{N}_0$ and

$$\sum_{k=0}^{\infty} \left| \frac{1}{2} a^{[k]}(1) - 1 \right| < \infty. \quad (4)$$

Then the infinite product

$$\prod_{k=1}^{\infty} \frac{1}{2} a^{[k-1]}(e^{i\omega 2^{-k}})$$

converges uniformly on compact subsets of \mathbb{R} .

Note that by Remark 3 the basic limit function Φ_0 of a convergent subdivision scheme satisfying (i) in Proposition 6 has compact support. Hence the continuous function Φ_0 is integrable and square integrable. It follows that the Fourier transform $\widehat{\Phi}_0$ of Φ_0 is well-defined, continuous and square integrable.

Proposition 7 *Let S_0 be a convergent subdivision scheme with symbols $a^{[k]}(z)$ for $k \in \mathbb{N}_0$ satisfying (i) and (ii) in Proposition 6. Then*

$$\widehat{\Phi}_0(\omega) = \prod_{k=1}^{\infty} \frac{1}{2} a^{[k-1]}(e^{i\omega 2^{-k}}).$$

Definition 8 *A subdivision schemes S_0 with masks $a_j^{[k]}, j \in \mathbb{Z}, k \in \mathbb{N}_0$ **reproduces a continuous function** $f : \mathbb{R} \rightarrow \mathbb{C}$ at level $k \in \mathbb{N}_0$ if for all $j \in \mathbb{Z}$*

$$f\left(\frac{j}{2^{k+1}}\right) = \sum_{l \in \mathbb{Z}} a_{j-2l}^{[k]} f\left(\frac{l}{2^k}\right). \quad (5)$$

*We say that S_0 **reproduces** f **stepwise** if it reproduces f at each level $k \in \mathbb{N}_0$.*

In [15, p. 31, just called reproducing] this is called just reproducing, and in [12, here "stepwise"] is defined the stepwise reproduction.

Proposition 9 Let S_0 be a subdivision scheme with symbols $a^{[k]}(z)$ for $k \in \mathbb{N}_0$, let $\lambda \in \mathbb{C}$, and $g(x)$ be a polynomial. Define $z^{[k]} := \exp(-2^{-(k+1)}\lambda)$, and for $\delta = 0, 1$ define

$$F_\delta^{[k]}(g, x) := \sum_{l \in \mathbb{Z}} a_{\delta-2l}^{[k]} g\left(\frac{l+x}{2^k}\right) (z^{[k]})^{\delta-2l}$$

for $\delta = 0, 1$.

Then for fixed $k \in \mathbb{N}_0$ the following statements are equivalent :

- a) $g(x) e^{\lambda x}$ is reproduced at level k .
- b) $F_\delta^{[k]}(g, m) = g\left(\frac{2m+\delta}{2^{k+1}}\right)$ for $\delta = 0, 1$ and for all $m \in \mathbb{Z}$.
- c) $F_\delta^{[k]}(g, x) = g\left(\frac{2x+\delta}{2^{k+1}}\right)$ for $\delta = 0, 1$ and for all $x \in \mathbb{R}$.

Proof. Using (5) for $j = 2m + \delta$ with $m \in \mathbb{Z}$ and $\delta = 0, 1$, it is easy to see that the function $g(x) e^{\lambda x}$ is reproduced stepwise if and only if for $\delta = 0, 1$ and for all $m \in \mathbb{Z}$

$$g\left(\frac{2m+\delta}{2^{k+1}}\right) = \sum_{l \in \mathbb{Z}} a_{2m+\delta-2l}^{[k]} g\left(\frac{l}{2^k}\right) (z^{[k]})^{2m+\delta-2l}.$$

Using the variable transformation $\tilde{l} = l + m$ it is easy to see that the right hand side is equal to $F_\delta(g, m)$, hence the equivalence of a) and b) is proven. The equivalence of b) and c) follows from the fact that $g(x)$ and $F_\delta(g, x)$ are polynomials. ■

Theorem 10 below extends Theorem 2.3 in [14] which was formulated only for interpolatory schemes. (We will apply this later to Daubechies schemes which are not interpolatory).

Theorem 10 Let S_0 be a subdivision scheme with masks $a_j^{[k]}, j \in \mathbb{Z}, k \in \mathbb{N}_0$. Then for $r = 0, \dots, \mu - 1$ the functions $f_r(x) = x^r e^{\lambda x}$ are reproduced stepwise by S_0 if and only if for all $k \in \mathbb{N}_0$ holds

$$a^{[k]}(-\exp(-2^{-(k+1)}\lambda)) = 0 \text{ and } a^{[k]}(\exp(-2^{-(k+1)}\lambda)) = 2 \quad (6)$$

and

$$\frac{d^r}{dz^r} a^{[k]}(\pm \exp(-2^{-(k+1)}\lambda)) = 0, \quad r = 1, \dots, \mu - 1. \quad (7)$$

Proof. Put $z^{[k]} := \exp(-2^{-(k+1)}\lambda)$. Since $F_\delta^{[k]}(g_\delta, 0) = \sum_{l \in \mathbb{Z}} a_{\delta-2l}^{[k]} g_\delta\left(\frac{l}{2^k}\right) (z^{[k]})^{\delta-2l}$, we first observe the following identities for the constant function $g = 1$:

$$a^{[k]}(z^{[k]}) = F_0^{[k]}(1, 0) + F_1^{[k]}(1, 0), \quad (8)$$

$$a^{[k]}(-z^{[k]}) = F_0^{[k]}(1, 0) - F_1^{[k]}(1, 0). \quad (9)$$

Suppose S_0 reproduces the function $f(x) = g(x) e^{\lambda x}$ where $g(x)$ is a polynomial of degree $\leq \mu - 1$. We obtain from condition b) in Proposition 9 for $g = 1$ that $1 = F_\delta^{[k]}(1, 0)$ for

$\delta = 0, 1$, and we conclude from (8) and (9) that $a^{[k]}(z^{[k]}) = 2$ and $a^{[k]}(-z^{[k]}) = 0$. Next we take

$$g_\delta^s(x) = (\delta - 2^{k+1}x)(\delta - 2^{k+1}x - 1) \dots (\delta - 2^{k+1}x - (s-1)) \quad (10)$$

for $s = 1, \dots, \mu - 1$. Then

$$g_\delta^s\left(\frac{l}{2^k}\right) = (\delta - 2l)(\delta - 2l - 1) \dots (\delta - 2l - (s-1)). \quad (11)$$

Since $F_\delta^{[k]}(g_\delta, 0) = \sum_{l \in \mathbb{Z}} a_{\delta-2l}^{[k]} g_\delta\left(\frac{l}{2^k}\right) (z^{[k]})^{\delta-2l}$ it follows from (11) that

$$\frac{d^s}{dz^s} a^{[k]}(z^{[k]}) = F_0^{[k]}(g_0^s, 0) + F_1^{[k]}(g_1^s, 0), \quad (12)$$

$$\frac{d^s}{dz^s} a^{[k]}(-z^{[k]}) = F_0^{[k]}(g_0^s, 0) - F_1^{[k]}(g_1^s, 0). \quad (13)$$

Since $g_\delta^s(x) e^{\lambda x}$ is reproduced, we obtain that $F_\delta^{[k]}(g_\delta^s, 0) = g_\delta\left(\frac{\delta}{2^{k+1}}\right) = 0$ for $\delta = 0, 1$. Thus (7) holds and the necessity part is proved.

The converse is proved by induction over the dimension s of the linear space E_s of all polynomials g of degree $\leq s$. By Proposition 9 we have to prove that for each $g \in E_s$ holds

$$g\left(\frac{2m + \delta}{2^{k+1}}\right) = F_\delta^{[k]}(g, m) \text{ for } \delta = 0, 1 \text{ and for all } m \in \mathbb{Z}.$$

For $s = 0$ this means we have to prove that $F_\delta^{[k]}(1, m) = 1$ for the constant function $g = 1$, for $\delta = 0, 1$, and for all $m \in \mathbb{Z}$. Since $F_\delta^{[k]}(g, x)$ is the constant polynomial we have only to show that $F_\delta^{[k]}(1, 0) = 1$. By assumption (7) and equations (8) and (9) we infer $2 = a^{[k]}(z^{[k]}) = F_0^{[k]}(1, 0) + F_1^{[k]}(1, 0)$ and $0 = a^{[k]}(-z^{[k]}) = F_0^{[k]}(1, 0) - F_1^{[k]}(1, 0)$, from which we infer the desired statement $F_\delta^{[k]}(1, 0) = 1$ for $\delta = 0, 1$.

Suppose that the statement is proven for E_{s-1} . We consider the polynomial g_δ^s defined in (10) for $\delta = 0, 1$. Note that equation (12) and (13), and the assumption (7) imply that

$$F_\delta^{[k]}(g_\delta^s, 0) = 0 = g_\delta^s\left(\frac{\delta}{2^{k+1}}\right) \quad (14)$$

for all $s = 1, \dots, \mu - 1$. Put now $g^s := g_0^s$. Then by (14) it follows $F_0^{[k]}(g^s, 0) = g^s(0)$. Note that

$$g_1^s(x) = g_0^s(x) + G^{s-1}(x)$$

for some polynomial G^{s-1} of degree $\leq s - 1$. Then

$$F_1^{[k]}(g_0^s, 0) = F_1^{[k]}(g_1^s, 0) - F_1^{[k]}(G^{s-1}, 0) = -G^{s-1}\left(\frac{1}{2^{k+1}}\right) = g_0^s\left(\frac{1}{2^{k+1}}\right),$$

since $F_1^{[k]}(g_1^s, 0) = 0$ by (14), and $F_1^{[k]}(G^{s-1}, 0) = G^{s-1}(\frac{1}{2^{k+1}})$ by the induction hypothesis. Hence

$$F_\delta^{[k]}(g^s, 0) = g^s\left(\frac{\delta}{2^{k+1}}\right).$$

If we prove that $F_\delta^{[k]}(g^s, m) = g^s(\frac{2m+\delta}{2^{k+1}})$ for all $m \in \mathbb{Z}$ and $\delta = 0, 1$, then we can conclude that g^s is reproduced by the scheme and clearly this implies that E_s is reproduced. Note that for each $m \in \mathbb{Z}$ there exists a polynomial h_m^s of degree $< s$ such that

$$g^s\left(x + \frac{m}{2^k}\right) = g^s(x) + h_m^s(x).$$

Then $g^s(\frac{l}{2^k} + \frac{m}{2^k}) = g^s(\frac{l}{2^k}) + h_m^s(\frac{l}{2^k})$ and

$$F_\delta^{[k]}(g^s, m) = \sum_{l \in \mathbb{Z}} a_{\delta-2l}^{[k]} g^s\left(\frac{l+m}{2^k}\right) (z^{[k]})^{\delta-2l} = F_\delta^{[k]}(g^s, 0) + F_\delta^{[k]}(h_m^s, 0).$$

By the induction hypothesis we know that $F_\delta^{[k]}(h_m^s, 0) = h_m^s(\frac{\delta}{2^{k+1}})$. It follows from (14) that

$$F_\delta^{[k]}(g^s, m) = g^s\left(\frac{\delta}{2^{k+1}}\right) + h_m^s\left(\frac{\delta}{2^{k+1}}\right) = g^s\left(\frac{\delta + 2m}{2^{k+1}}\right).$$

The proof is complete. ■

It is easy to see that a function $f : \mathbb{R} \rightarrow \mathbb{C}$ is reproduced stepwise by a subdivision scheme if and only if for the data function $f^{[0]}(j) := f(j)$ one has

$$f^{[k]}\left(\frac{j}{2^k}\right) = f\left(\frac{j}{2^k}\right).$$

From this it is easy to see that a convergent and stepwise reproducing subdivision scheme is reproducing in the following sense:

Definition 11 *A convergent subdivision scheme S_0 with masks $a_j^{[k]}, j \in \mathbb{Z}, k \in \mathbb{N}_0$, is reproducing a continuous function $f : \mathbb{R} \rightarrow \mathbb{C}$ if the limit function for the data function $f(j), j \in \mathbb{Z}$, is equal to $f(x)$.*

Remark 12 *In the definition of a convergent scheme some authors require bounded data.*

An important concept for the investigation of non-stationary subdivision schemes is the following notion introduced in [16]:

Definition 13 *Two subdivision schemes S_a and S_b with masks $a_j^{[k]}, j \in \mathbb{Z}, k \in \mathbb{N}_0$, and $b_j^{[k]}, j \in \mathbb{Z}, k \in \mathbb{N}_0$, resp. are called **asymptotically equivalent** if*

$$\sum_{k=0}^{\infty} \sum_{j \in \mathbb{Z}} |a_j^{[k]} - b_j^{[k]}| < \infty. \quad (15)$$

We say that S_a and S_b are **exponentially asymptotically equivalent** if there exists a constant $C > 0$ such that

$$\max_{j \in \mathbb{Z}} |a_j^{[k]} - b_j^{[k]}| \leq C \cdot 2^{-k} \quad (16)$$

for all $k \in \mathbb{N}_0$.

We also say that the masks $a^{[k]}$, $k \in \mathbb{N}_0$, and $b^{[k]}$, $k \in \mathbb{N}_0$, are (exponentially resp.) asymptotically equivalent if (15) ((16) resp.) holds.

Suppose that the masks $a_j^{[k]}$, $j \in \mathbb{Z}$, $k \in \mathbb{N}_0$, and $b_j^{[k]}$, $j \in \mathbb{Z}$, $k \in \mathbb{N}_0$, have support in the set $\{-N, \dots, N\}$, i.e. $a_j^{[k]} = 0$ for $|j| > N$. Then it is easy to see that $a^{[k]}$, $k \in \mathbb{N}_0$, and $b^{[k]}$, $k \in \mathbb{N}_0$, are exponentially asymptotically equivalent if and only if for any $R > 1$ there exists $D > 0$ such that

$$|a^{[k]}(z) - b^{[k]}(z)| \leq D \cdot 2^{-k}$$

for all $k \in \mathbb{N}_0$ and for all $z \in \mathbb{C}$ with $1/R \leq |z| \leq R$.

The following result provides a sufficient method for constructing asymptotically equivalent subdivision schemes.

Theorem 14 *Let $m \in \mathbb{N}_0$ and assume that $p^{[k]}(z)$ and $p(z)$ are polynomials of degree m for each $k \in \mathbb{N}_0$ defined as*

$$p^{[k]}(z) = c^{[k]} \prod_{j=1}^m (z - \alpha_j^{[k]}) \quad \text{and} \quad p(z) = c \prod_{j=1}^m (z - \alpha_j)$$

for some complex numbers $c^{[k]}$, c , and $\alpha_j^{[k]}$ and α_j , for $j = 1, \dots, m$, and $k \in \mathbb{N}_0$. Suppose that there exists a constant $D_m > 0$ such that for all $k \in \mathbb{N}_0$ and $j = 1, \dots, m$

$$|\alpha_j^{[k]} - \alpha_j| \leq D_m 2^{-k} \quad \text{and} \quad |c^{[k]} - c| \leq D_m 2^{-k}. \quad (17)$$

Then $p^{[k]}(z)$, $k \in \mathbb{N}_0$, and $p(z)$, $k \in \mathbb{N}_0$, are exponentially asymptotically equivalent.

Proof. We claim by induction over $m \in \mathbb{N}_0$ that for each $R > 0$ there exists a constant $C_m(R)$ such that for all $k \in \mathbb{N}_0$ and $|z| \leq R$

$$|p^{[k]}(z) - p(z)| \leq C_m(R) 2^{-k}. \quad (18)$$

For $m = 0$ the statement follows directly from (17). Suppose that the statement is true for $m - 1$. Write

$$p^{[k]}(z) = c^{[k]} (z - \alpha_m^{[k]}) p_{m-1}^{[k]}(z) \quad \text{and} \quad p(z) = c (z - \alpha_m) p_{m-1}(z),$$

where $p_{m-1}^{[k]}(z)$ and $p_{m-1}(z)$ are polynomials of degree $\leq m-1$ and leading coefficient 1. By the induction hypothesis, for each $R > 0$ a constant $C_m(R)$ such that $|c^{[k]}p_{m-1}^{[k]}(z) - cp_{m-1}(z)| \leq C_m(R)2^{-k}$ for all $k \in \mathbb{N}_0$ and $|z| \leq R$. Note that

$$\begin{aligned} p^{[k]}(z) - p(z) &= c^{[k]}(z - \alpha_m^{[k]})p_{m-1}^{[k]}(z) - c(z - \alpha_m)p_{m-1}(z) \\ &= (z - \alpha_m) \left[c^{[k]}p_{m-1}^{[k]}(z) - cp_{m-1}(z) \right] + c^{[k]}(\alpha_m - \alpha_m^{[k]})p_{m-1}^{[k]}(z). \end{aligned}$$

Using the triangle inequality, the induction hypothesis, and (17), one obtains (18). ■

Theorem 15 *Assume that $p^{[k]}(z)$ and $p(z)$ are polynomials of degree $\leq m$ for each $k \in \mathbb{N}_0$ and assume for each $R > 0$ there exists a constant $C_m(R)$ such that for all $k \in \mathbb{N}_0$ and $|z| \leq R$*

$$|p^{[k]}(z) - p(z)| \leq C_m(R)2^{-k}. \quad (19)$$

If α is a simple zero of $p(z)$ then there exists $k_0 \in \mathbb{N}_0$ and a constant $\rho > 0$ such that for each natural number $k \geq k_0$ there exists a zero $\alpha^{[k]}$ of $p^{[k]}(z)$ with

$$|\alpha^{[k]} - \alpha| \leq \rho 2^{-k} \text{ for all } k \geq k_0.$$

Proof. Let $\rho > 0$ and define $\gamma_k(t) = \alpha + 2^{-k}\rho e^{it}$ for $t \in [0, 2\pi]$. Write $p(z) = \sum_{j=1}^m p_j(z - \alpha)^j$. Since $p(z)$ has a simple zero in α we know that $p_0 = 0$ and $p_1 \neq 0$. We obtain the estimate

$$|p(\gamma_k(t))| \geq \rho 2^{-k} \left(|p_1| - \sum_{j=2}^m |p_j| \frac{\rho^j}{2^{k(j-1)}} \right).$$

Hence for given $\rho > 0$ there exists a natural number k_ρ such that for all $k \geq k_\rho$

$$|p(\gamma_k(t))| \geq \rho 2^{-k} \frac{|p_1|}{2}$$

and for all $t \in [0, 2\pi]$. Now take $R > 0$ is large enough that $R > 2|\alpha|$. Take ρ such that $\rho \frac{|p_1|}{2} > C_m(R)$ where $C_m(R)$ is the constant in (19). Take $k_1 \geq k_\rho$ large enough that $|\gamma_k(t)| \leq R$ for all $k \geq k_1$ and $t \in [0, 2\pi]$. Then

$$|p^{[k]}(\gamma_k(t)) - p(\gamma_k(t))| \leq C_m(R)2^{-k} < \rho \frac{|p_1|}{2} 2^{-k} < |p(\gamma_k(t))|$$

for all $k \geq k_1$ and $t \in [0, 2\pi]$. By Rouché's theorem the number of zeros of $p^{[k]}(z)$ in the ball $|z - \alpha| < 2^{-k}\rho$ is equal to the number of zeros of $p(z)$ in that ball. Since $p(\alpha) = 0$ it follows that for each $k \geq k_1$ there exists a zero $\alpha^{[k]}$ of $p^{[k]}(z)$ in the ball $|z - \alpha| < 2^{-k}\rho$, hence, $|\alpha^{[k]} - \alpha| < 2^{-k}\rho$. ■

3 Subdivision schemes based on exponential interpolation and regularity of the basic limit function

Let us first recall some basic facts about the classical $2n$ -point Deslauriers-Dubuc subdivision scheme.

It is defined via interpolation of polynomials of degree $2n - 1$, see [9]. We shall denote its symbol by $D_{2n}(z)$ which is given by

$$D_{2n}(z) = \sum_{|j| \leq 2n-1} p_j z^j. \quad (20)$$

According to Theorem 6.1 in [9] the symbol $D_{2n}(z)$ has the (reproduction) property if and only if

$$\frac{d^j}{dz^j} D_{2n}(-1) = 0, \text{ for } j = 0, \dots, 2n - 1. \quad (21)$$

Further the scheme of Deslauriers and Dubuc is interpolatory, i.e., $p_{2j} = \delta_{0,j}$, for all $j \in \mathbb{Z}$, or equivalently

$$D_{2n}(z) + D_{2n}(-z) = 2, \text{ for all } z \in \mathbb{C} \setminus \{0\}. \quad (22)$$

Together, conditions (21) and (22) constitute a linear system which uniquely determines the symbol $D_{2n}(z)$ of the Deslauriers and Dubuc scheme and it can be written in the form

$$D_{2n}(z) = \left(\frac{1+z}{2} \right)^{2n} b_{D_{2n}}(z). \quad (23)$$

Let us mention that condition (21) means that polynomials of degree $\leq 2n - 1$ are reproduced by the subdivision scheme. The Laurent polynomial $b_{D_{2n}}(z)$ can be computed explicitly and the following identity (see e.g. [13]),

$$D_{2n}(z) = 2 \frac{(1+z)^n \left(1 + \frac{1}{z}\right)^n}{2^{2n}} Q_{n-1}(\varphi(z)) \quad (24)$$

holds for all $z \neq 0$, where $\varphi(z) = \frac{1}{2} - \frac{1}{4}(z + 1/z)$ and $Q_{n-1}(x)$ is the polynomial of degree $n - 1$ given by

$$Q_{n-1}(x) = \sum_{j=0}^{n-1} \binom{n+j-1}{j} x^j = \sum_{j=0}^{n-1} \frac{(n+j-1)!}{j!(n-1)!} x^j. \quad (25)$$

The next result shows that $Q_{n-1}(z)$ and $b_{D_{2n}}(z)$ have only simple zeros.

Proposition 16 *The polynomial $Q_{n-1}(x)$ in (25) satisfies the identity*

$$nQ_{n-1}(x) + (x-1)Q'_{n-1}(x) = \frac{(2n-1)!}{(n-1)!(n-1)!} x^{n-1}.$$

Proof. Clearly $Q'_{n-1}(x) = \sum_{j=0}^{n-2} \frac{(n+j)!}{j!(n-1)!} x^j$. Then $nQ_{n-1}(x) + (x-1)Q'_{n-1}(x)$ is equal to

$$n \sum_{j=0}^{n-1} \frac{(n+j-1)!}{j!(n-1)!} x^j + \sum_{j=1}^{n-1} \frac{(n+j-1)!}{(j-1)!(n-1)!} x^j - \sum_{j=0}^{n-2} \frac{(n+j)!}{j!(n-1)!} x^j.$$

from which the statement follows. ■

Now we turn to subdivision schemes for exponential polynomials. Let L be the linear differential operator given by

$$L = \left(\frac{d}{dx} - \lambda_0 \right) \dots \left(\frac{d}{dx} - \lambda_n \right). \quad (26)$$

Complex-valued solutions f of the equation $Lf = 0$ are called *L-polynomials* or *exponential polynomials* or just *exponentials*. We shall denote the set of all solutions of $Lf = 0$ by $E(\lambda_0, \dots, \lambda_n)$.

Throughout this article we shall assume that $\lambda_0, \dots, \lambda_n$ are real numbers. This assumption implies that $E(\lambda_0, \dots, \lambda_n)$ is an **extended Chebyshev space**, in particular for any pairwise distinct points t_0, \dots, t_n and data values y_0, \dots, y_n , there exists a unique element $p \in E(\lambda_0, \dots, \lambda_n)$ with $p(t_j) = y_j$ for $j = 0, \dots, n$, i.e. p is interpolating exponential polynomial, cf. [26].

Given real numbers $\lambda_0, \dots, \lambda_{2n-1}$ one can define the *subdivision scheme based on interpolation in $E(\lambda_0, \dots, \lambda_{2n-1})$* : the new value $f^{k+1}(j/2^{k+1})$ is computed by constructing the unique function $p_j \in E(\lambda_0, \dots, \lambda_{2n-1})$ interpolating the previous data $f^k((j+l)/2^k)$ for $l = -n+1, \dots, n$, and putting $f^{k+1}(j/2^{k+1}) = p_j(j/2^{k+1})$ (see [30] for details). Then the symbols of this scheme are of the form

$$a^{[k]}(z) = \sum_{|j| \leq 2n-1} a_j^{[k]} z^j \quad (27)$$

and since the scheme is interpolatory one has

$$a^{[k]}(z) + a^{[k]}(-z) = 2, \quad z \in \mathbb{C} \setminus \{0\}. \quad (28)$$

Due to the interpolatory definition of the subdivision it is clear that each function $f \in E(\lambda_0, \dots, \lambda_{2n-1})$ is reproduced stepwise by the scheme, and Theorem 10 implies that

$$\frac{d^s}{dz^s} a^{[k]}(-\exp(-2^{-(k+1)} \lambda_j)) = 0, \quad s = 0, \dots, \mu_j - 1. \quad (29)$$

where μ_j is the *multiplicity* of λ_j , i.e. the number of times the value λ_j occurs in $(\lambda_0, \dots, \lambda_{2n-1})$. Hence the subdivision scheme based on interpolation in $E(\lambda_0, \dots, \lambda_{2n-1})$ is completely characterized by (28) and (29). In the terminology of [14] this is the *even-order, symmetric and minimal rank scheme reproducing $E(\lambda_0, \dots, \lambda_{2n-1})$* . Note that the Deslauriers-Dubuc scheme is a special case by taking $\lambda_0 = \dots = \lambda_{2n-1} = 0$, reproducing the space of algebraic polynomials Π_{2n-1} .

Definition 17 For given real numbers $\lambda_0, \dots, \lambda_{2n-1}$ let $a^{[k]}, k \in \mathbb{N}_0$, be the symbols satisfying (28) and (29). For any natural number $m \in \mathbb{N}_0$ we define a new subdivision scheme $S_m^{\Lambda_{2n}}$ defined by the symbols

$$a^{[k],m}(z) = a^{[k+m]}(z) \text{ for } k \in \mathbb{N}_0.$$

In the rest of the paper we shall use the notations given in Definition 17.

Remark 18 It is easy to see that the subdivision scheme $S_m^{\Lambda_{2n}}$ is again an even-order, symmetric and minimal rank scheme reproducing $E(\lambda_0/2^m, \dots, \lambda_{2n-1}/2^m)$.

Many properties of the subdivision scheme $S_m^{\Lambda_{2n}}$ for exponential polynomials can be derived from their polynomial counterpart, the Deslauriers-Dubuc scheme. The key to these results depend on the following observation in [14, Theorem 2.7].

Proposition 19 The subdivision scheme $S_m^{\Lambda_{2n}}$ is exponentially asymptotically equivalent to the $2n$ -point Deslauriers-Dubuc subdivision scheme, i.e. there exists $C > 0$ such that

$$\sum_{|j| \leq 2n-1} |p_j - a_j^{[k],m}| \leq C 2^{-k} \text{ for all } k \in \mathbb{N}_0. \quad (30)$$

Deslauriers and Dubuc showed in [9] that their scheme is C^0 -convergent implying the existence of a basic limit function which will be denoted in the following by $\Phi^{D_{2n}}$. Furthermore, one can find sufficient criteria for the C^ℓ -convergence in [9].

According to Theorem 2.10 in [14] we have the following result

Theorem 20 Let $\ell \in \mathbb{N}_0$. If the Deslauriers-Dubuc subdivision scheme is C^ℓ -convergent then the subdivision scheme $S_m^{\Lambda_{2n}}$ is C^ℓ -convergent as well.

According to the last theorem the subdivision scheme $S_m^{\Lambda_{2n}}$ has a basic limit function which will be denoted in the following by $\Phi_m^{\Lambda_{2n}}$.

A function $f : \mathbb{R}^d \rightarrow \mathbb{C}$ is called *Lipschitz function of order α* (or *Hölder function of order α*) if there exists a number $L > 0$ such that for all $x, y \in \mathbb{R}^d$

$$|f(x) - f(y)| \leq L |x - y|^\alpha.$$

The set of all Lipschitz functions of order α is denoted by $Lip(\alpha)$. Functions in $Lip(\alpha)$ can be characterized via Fourier transform and we just recall Lemma 7.1 in [9, p. 56]:

Lemma 21 Let $f : \mathbb{R} \rightarrow \mathbb{C}$ be an integrable function whose Fourier transform is $g(\xi)$. We assume that $|\xi|^{\ell+\alpha} g(\xi)$ is integrable where $\ell \in \mathbb{N}_0$ and $\alpha \in [0, 1]$. If $\alpha = 0$ then f is ℓ times continuously differentiable. If $\alpha \neq 0$, then $f^{(\ell)}$ is a Lipschitz function of order α .

Theorem 22 *Let $\alpha \in [0, 1)$ and $\ell \in \mathbb{N}_0$ and assume that the basic limit function $\Phi^{D_{2n}}$ of the $2n$ -point Deslauriers-Dubuc scheme satisfies for some $\varepsilon > 0$ and $C > 0$ the inequality*

$$\left| \widehat{\Phi^{D_{2n}}}(\omega) \right| \leq C (|\omega| + 1)^{-\ell-1-\alpha-\varepsilon} \quad (31)$$

for all $\omega \in \mathbb{R}$. Then the basic limit function $\Phi_m^{\Lambda_{2n}}$ of the scheme $S_m^{\Lambda_{2n}}$ has its ℓ -th derivative in $Lip(\alpha)$.

Proof. We begin with some general remarks: let us define $z_j^{[k]} = \exp(-2^{-(k+1)}\lambda_j)$. Using (29) we can write

$$a^{[k]}(z) = \left(\prod_{j=1}^{2n} \frac{z + z_j^{[k]}}{2} \right) b^{[k]}(z). \quad (32)$$

Let $D_{2n}(z) = \left(\frac{1+z}{2}\right)^{2n} b_{D_{2n}}(z)$ be the symbol of the Deslauriers-Dubuc scheme. It was shown in [14, Formula (2.32)] that $b^{[k]}$ is exponentially asymptotically equivalent to $b_{D_{2n}}$, so there exists a constant $B > 0$ such that

$$|b^{[k]}(e^{i\omega}) - b_{D_{2n}}(e^{i\omega})| \leq B \cdot 2^{-k} \quad (33)$$

for all $\omega \in \mathbb{R}$ and for all $k \in \mathbb{N}_0$. As an intermediate step we consider the non-stationary scheme S_m^c defined by the symbols

$$c_m^{[k]}(z) = \left(\frac{z+1}{2} \right)^{2n} b^{[m+k]}(z). \quad (34)$$

According to the proof of Theorem 2.10 in [14] the scheme S_m^c has a basic limit function denoted by Φ_m^c . Let $\Phi_m^{\Lambda_{2n}}$ be the basic limit function of $S_m^{\Lambda_{2n}}$. By Proposition 7

$$\widehat{\Phi_m^{\Lambda_{2n}}}(\omega) = \prod_{k=1}^{\infty} \frac{1}{2} a^{[m+k-1]}(e^{i\omega 2^{-k}}). \quad (35)$$

Similarly, $\widehat{\Phi^{D_{2n}}}(\omega) = \prod_{k=1}^{\infty} \frac{1}{2} a(e^{i\omega 2^{-k}})$ and we see that

$$\frac{\widehat{\Phi_m^c}(\omega)}{\widehat{\Phi^{D_{2n}}}(\omega)} = \prod_{k=1}^{\infty} \frac{b^{[m+k-1]}(e^{i\omega 2^{-k}})}{b_{D_{2n}}(e^{i\omega 2^{-k}})} \quad (36)$$

$$= \prod_{k=1}^{\infty} \left(1 + \frac{b^{[m+k-1]}(e^{i\omega 2^{-k}}) - b_{D_{2n}}(e^{i\omega 2^{-k}})}{b_{D_{2n}}(e^{i\omega 2^{-k}})} \right). \quad (37)$$

It is well known [9] that the trigonometric polynomial $b_{D_{2n}}(e^{-i\omega 2^{-k}})$ does not vanish on the unit circle, hence the denominator in (37) satisfies

$$\left| b_{D_{2n}}(e^{i\omega 2^{-k}}) \right| \geq \delta > 0 \quad (38)$$

for some $\delta > 0$. Using (38) and (33) it is straightforward to prove that the infinite product in (36) is uniformly bounded for $\omega \in \mathbb{R}$, and we obtain

$$\left| \frac{\widehat{\Phi_m^c}(\omega)}{\widehat{\Phi^{D_{2n}}}(\omega)} \right| \leq M_m .$$

Recalling that each factor $\frac{1+z}{2}$ induces convolution with a B -spline of order 0 (denoted as usually by B_0) the above inequality can be written as

$$\left| \widehat{\Phi_m^c}(\omega) \right| = \left(\frac{\sin \frac{\omega}{2}}{\frac{\omega}{2}} \right)^{2n} \prod_{k=1}^{\infty} \left| b^{[m+k-1]} \left(e^{i\omega 2^{-k}} \right) \right| \leq M_m \left| \widehat{\Phi^{D_{2n}}}(\omega) \right| \quad (39)$$

$$\leq M_n C(|\omega| + 1)^{-\ell-1-\alpha-\varepsilon} . \quad (40)$$

Comparing (32) and (34) we see that in order to prove the theorem we need to replace each factor $\frac{\sin \frac{\omega}{2}}{\frac{\omega}{2}}$ by the Fourier transform of the basic limit function generated by the subdivision scheme with symbols $\left\{ \frac{z+z_j^{[k]}}{2} \right\}$. This should be done with care since $\frac{\sin \frac{\omega}{2}}{\frac{\omega}{2}}$ vanishes at infinite number of points on \mathbb{R} . We employ another observation from [16], that the basic limit function Φ_0 of the scheme with symbols $\left\{ \frac{z+z_j^{[k]}}{2} \right\}$ is an exponential B -spline of order 0,

$$B_0^j(\omega) := \begin{cases} e^{\lambda_j x} & \text{for } x \in [0, 1] \\ 0 & \text{otherwise} \end{cases} .$$

Adding such factors, $\left\{ \frac{z+z_j^{[k]}}{2} \right\}_{j=1}^{2n}$ to the symbols $\{c^{[k]}\}$ results in repeated convolutions of $\widehat{\Phi_m^c}$ with $\{B_0^j\}_{j=1}^{2n}$. Since $\widehat{B_0^j}(x)$ decays as $\frac{1}{|x|}$ for $|\omega| \rightarrow \infty$, each convolution adds one power to the decay power α in (39). Hence, after $2n$ convolutions we obtain

$$\begin{aligned} \widehat{\Psi_m}(\omega) &:= \left(\frac{\sin \frac{\omega}{2}}{\frac{\omega}{2}} \right)^{2n} \prod_{k=1}^{\infty} a^{[m+k-1]} b \left(e^{-i\omega 2^{-k}} \right) \\ &= \left(\frac{\sin \frac{\omega}{2}}{\frac{\omega}{2}} \right)^{2n} \widehat{\Phi_m^{\Lambda_{2n}}}(\omega) = O \left((|\omega| + 1)^{-\ell-1-\alpha-2n} \right) . \end{aligned} \quad (41)$$

It follows by Lemma 21 that the $(2n + \ell)$ -th derivative of Ψ_m is $Lip(\alpha)$. Now we are ready to return to $\widehat{\Phi_m^{\Lambda_{2n}}}$ by removing the factors $\left(\frac{\sin \frac{\omega}{2}}{\frac{\omega}{2}} \right)^{2n}$ from $\widehat{\Psi_m}(\omega)$. We need to show that each factor removed implies that the order of the derivative which is in $Lip(\alpha)$ is reduced by one. To show this we consider $g = B_0 * f$, where f is a function of compact support and g' is in $Lip(\alpha)$. It follows that

$$g'(s) = f(s) - f(s-1).$$

Summing the above relations over all numbers $s \in \{t + j\}_{j=0}^N$, for large enough N , we obtain

$$f(t) = \sum_{j=0}^N g'(t + j) ,$$

implying that f is $Lip(\alpha)$. Since $\hat{g}(\omega) = \frac{\sin \frac{\omega}{2}}{\frac{\omega}{2}} \hat{f}(\omega)$, we have just shown that the consequence of removing a factor $\frac{\sin \frac{\omega}{2}}{\frac{\omega}{2}}$ is that the order of the derivative which is in $Lip(\alpha)$ is reduced by one. Removing $2n$ such factors yields the desired result, namely, that the ℓ -th derivative of $\{\Phi_m^{\Lambda_{2n}}\}$ is in $Lip(\alpha)$. ■

Remark 23 In [13], [20], [21] motivated by the multivariate polyharmonic subdivision and wavelets on parallel hyperplanes, an explicit expression for the polynomial $b^{[k]}(z)$ is found, for the case of frequencies given by $\lambda_j = \xi$, $j = 0, 1, \dots, n-1$, for some real $\xi \geq 0$, and $\lambda_j = -\xi$, for $j = n, \dots, 2n$. The classical Deslaurier-Dubuc case corresponds to $\xi = 0$.

4 Non-stationary multiresolutional analysis

The concept of a multi-resolutional analysis, introduced by S. Mallat and Y. Meyer, is an effective tool to construct wavelets in a simple way from a given scaling function φ , see e.g. [1], [6]. *Non-stationary multiresolution analysis* was introduced in [8] by C. de Boor, R. DeVore and A. Ron. For convenience of the reader we recall here the definition for the univariate case:

Definition 24 A (non-stationary) multiresolution analysis consists of a sequence of closed subspaces $V_m, m \in \mathbb{Z}$, in $L^2(\mathbb{R})$ satisfying

- (i) $V_m \subset V_{m+1}$ for all $m \in \mathbb{Z}$,
- (ii) the intersection $\cap_{m \in \mathbb{Z}} V_m$ is the trivial subspace $\{0\}$,
- (iii) the union $\cup_{m \in \mathbb{Z}} V_m$ is dense in $L^2(\mathbb{R})$,
- (iv) for each $m \in \mathbb{Z}$ there exists a function $\varphi_m \in V_m$ such that the family of functions $\{\varphi_m(2^m t - k) : k \in \mathbb{Z}\}$ form a Riesz basis of V_m .

The function φ_m in condition (iv) is called a *scaling function* for V_m . The requirement (iv) means that for each $f \in V_m$ there exists a unique sequence $(c_k)_{k \in \mathbb{Z}}$ in $l^2(\mathbb{Z})$ (i.e. that $\sum_{k=-\infty}^{\infty} |c_k|^2 < \infty$) such that

$$f(t) = \sum_{k=-\infty}^{\infty} c_k \varphi_m(2^m t - k)$$

with convergence in $L^2(\mathbb{R})$ and

$$A_m \sum_{k=-\infty}^{\infty} |c_k|^2 \leq \left\| \sum_{k=-\infty}^{\infty} c_k \varphi_m(2^m t - k) \right\|^2 \leq B_m \sum_{k=-\infty}^{\infty} |c_k|^2$$

for all $(c_k)_{k \in \mathbb{Z}}$ in $l^2(\mathbb{Z})$ with $0 < A_m \leq B_m < \infty$ constants independent of $f \in V_m$.

Using (i) one can define the *wavelet space* W_m as the unique subspace such that $V_m \oplus W_m = V_{m+1}$ for $m \in \mathbb{Z}$ and W_m is orthogonal to V_m . It easy to see that this implies that W_k and W_m are orthogonal subspaces for $k \neq m$. Conditions (ii) and (iii) imply that

$$L^2(\mathbb{R}) = \oplus_{m \in \mathbb{Z}} W_m.$$

We refer to [8] for an extensive discussion on the construction of so-called pre-wavelets for a nonstationary multiresolutional analysis, see also [36]. Important examples of nonstationary multiresolutions occur in the context of cardinal exponential-splines wavelets which generalizes the work of C.K. Chui and J.Z. Wang about cardinal spline wavelets in [4], see [2], [3]. The interested reader may consult [29], [32] for the theory of exponential splines and [8], [19], [24], [25], [28] for the construction of wavelets in this context. In passing we mention that the results in the last cited papers have been rediscovered in [35].

Definition 25 *A multiresolutional analysis is called stationary if in condition (iv) the scaling function $\varphi_m \in V$ is the same for all $m \in \mathbb{Z}$.*

In this paper we shall be concerned only with orthonormal non-stationary MRA:

Definition 26 *A multiresolutional analysis is called orthonormal if in condition (iv) the functions $t \mapsto 2^{m/2} \varphi_m(2^m t - k)$ for $k \in \mathbb{Z}$ are an orthonormal basis of V_m .*

Let us recall that an *orthonormal wavelet* ψ is a function in $L^2(\mathbb{R})$ such that the system $\psi_{m,k}(x) = 2^{m/2} \psi(2^m x - k)$ with $m, k \in \mathbb{Z}$ is an orthonormal basis of $L^2(\mathbb{R})$. In the context of nonstationary wavelet analysis one wants to find a sequence of functions $\psi_m \in L^2(\mathbb{R})$, $m \in \mathbb{Z}$, such that

$$\psi_{m,k}(x) = 2^{m/2} \psi_m(2^m x - k)$$

with $m, k \in \mathbb{Z}$ is an orthonormal basis of $L^2(\mathbb{R})$.

5 Daubechies type wavelets

Daubechies wavelets ψ are orthonormal wavelets with compact support and certain degree of smoothness. The construction of Daubechies wavelets is often presented in the following way (see e.g. [1], [17], [27]) : using the concept of an orthonormal MRA it suffices to construct a suitable scaling function φ . Elementary considerations show that the Fourier transform $\widehat{\varphi}$ of the scaling function φ should be of the form

$$\widehat{\varphi}(\omega) = \prod_{k=1}^{\infty} m(2^{-k}\omega)$$

where $m(\omega)$ is a trigonometric polynomial with real coefficients and $m(0) = 1$ satisfying the equation

$$|m(\omega)|^2 + |m(\omega + \pi)|^2 = 1. \quad (42)$$

This leads to the question which *non-negative* trigonometric polynomials $q(\omega)$ satisfy an equation of the type

$$q(\omega) + q(\omega + \pi) = 1 \text{ and } q(0) = 1. \quad (43)$$

There are many explicit solutions of (43). For example, if n is a natural number then the trigonometric polynomial

$$q_n(\omega) = 1 - c_n \int_0^\omega (\sin t)^{2n-1} dt$$

with $c_n := \int_0^\pi \sin^{2n-1} t dt$ satisfies equation (43). By the Fejér-Riesz lemma one can find a (non-unique) trigonometric polynomial $m(\omega)$ such that

$$q_n(\omega) = |m(\omega)|^2. \quad (44)$$

We call a Laurent polynomial $m(\omega)$ with real coefficients and $m(0) = 1$ satisfying (44) a *Daubechies filter of order n* . The *Daubechies scaling function φ^m for the Daubechies filter $m(\omega)$* is then defined by

$$\widehat{\varphi^m}(\omega) = \prod_{k=1}^{\infty} m(2^{-k}\omega).$$

This procedure is elegant but the construction of the trigonometric polynomial q_n in the above approach seems to be rather miraculous. Let us emphasize that I. Daubechies has shown much more (see e.g. [6, p. 210] or [37]): the regularity of the wavelet and the scaling function imply that the symbol $m(\omega)$ must contain a factor $(1 + e^{i\omega})^n / 2^n$. Hence $q_n(\omega)$ is of the form

$$q_n(\omega) = \frac{(1 + e^{i\omega})^{2n}}{2^n} F_{2n-1}(\omega)$$

where $F_{2n-1}(\omega)$ is a suitable trigonometric polynomial with real coefficients which can be determined by Bezout's theorem from (43). Indeed, it follows from these considerations that

$$q_n(\omega) = D_{2n}(e^{i\omega})$$

where D_{2n} is the symbol of the Deslauriers-Dubuc subdivision scheme, a fact which is already mentioned by Daubechies in her book [6, Section 6.5] giving credit to this observation to M.J. Shensa in [33], see [6, p. 210]. *Hence the Deslauriers-Dubuc scheme leads in a very natural and direct way to the construction of the Daubechies scaling function and therefore, by MRA-methods, to Daubechies wavelets.*

In this section we want to use this concept in order to define Daubechies type wavelets for exponential polynomials. In this setting we have some additional freedom which is interesting for applications: we may choose real numbers $\lambda_0, \dots, \lambda_{n-1}$ and we shall construct *Daubechies type wavelets reconstructing the space $E(\lambda_0, \dots, \lambda_{n-1})$* . In the case of

Daubechies wavelets this corresponds to the fact that the Daubechies wavelet reproduces polynomials of degree $\leq n - 1$.

We shall write shortly $\Lambda_0 = (\lambda_0, \dots, \lambda_{n-1})$ and define $\lambda_{n+j} := -\lambda_j$ for $j = 0, \dots, n - 1$. We consider now the subdivision scheme based on interpolation in $E(\lambda_0, \dots, \lambda_{2n-1})$. According to Definition 17 the subdivision scheme $S_0^{\Lambda_{2n}}$ has the symbols

$$a^{[k]}(z) = \prod_{j=0}^{2n-1} \frac{z + z_j^{[k]}}{2} b^{[k]}(z) \text{ with } z_j^{[k]} := \exp(-2^{-(k+1)}\lambda_j) \quad (45)$$

for $k \in \mathbb{N}_0$ and $j = 0, \dots, 2n - 1$. Crucial is the result by Micchelli in [30] (Proposition 5.1), proving that the symbols $a^{[k]}$ satisfy

$$a^{[k]}(z) \geq 0, \text{ for } |z| = 1, \quad (46)$$

with equality possible only if $z = -1$. By the Fejer-Riesz lemma we can find for each level k a "square root" Laurent polynomial $M^{[k]}(z)$ with *real* coefficients, satisfying

$$a^{[k]}(e^{i\omega}) = \frac{1}{2} |M^{[k]}(e^{i\omega})|^2 \text{ and } M^{[k]}(1) > 0. \quad (47)$$

Note that this implies that

$$a^{[k]}(z) = \frac{1}{2} M^{[k]}(z) M^{[k]}(\frac{1}{z})$$

for all complex $z \neq 0$. Again, there are many Laurent polynomials $M^{[k]}(z)$ which satisfy (47) and all possible choices can be described through suitable subsets of the zero-set of $a^{[k]}(z)$. First we choose the roots $z = -\exp(-\lambda_j/2^{k+1})$ for $j = 0, \dots, n - 1$, in order to obtain stepwise reproduction of the space $E(\lambda_0, \dots, \lambda_{n-1})$, see Proposition 27 below. Further, we have to choose another $n - 1$ roots of the factor $b^{[k]}$ in (45). Since $b^{[k]}$ is symmetric, its $2n - 2$ roots come in inverse pairs, say z_i and z_i^{-1} , and as well complex conjugates \bar{z}_i and \bar{z}_i^{-1} if z_i is not real, for i in an index set I_{n-1} . We choose either the set $\{z_i, \bar{z}_i\}$ or the set $\{z_i^{-1}, \bar{z}_i^{-1}\}$ for each $i \in I_{n-1}$, leading to a Laurent polynomial with real coefficients which still has to be normalized so that $M^{[k]}(1) = \sqrt{2a^{[k]}(1)} > 0$. We shall call a sequence of filters $M^{[k]}(z)$, $k \in \mathbb{N}_0$, chosen in this way a *non-stationary Daubechies type subdivision scheme of order n* .

Since $M^{[k]}(1)$ is positive it follows that $1 \leq \frac{1}{2}M^{[k]}(1) + 1$ and $a^{[k]}(1) = \frac{1}{2}M^{[k]}(1)^2$ we infer that

$$\left| \frac{1}{2}M^{[k]}(1) - 1 \right| \leq \left| \frac{1}{2}M^{[k]}(1) - 1 \right| \left| \frac{1}{2}M^{[k]}(1) + 1 \right| = \left| \frac{1}{2}a^{[k]}(1) - 1 \right|.$$

Since $a^{[k]}(z)$ is exponentially asymptotically equivalent to the Deslauriers-Dubuc scheme and $\frac{1}{2}D_{2n}(1) = 1$ we infer that there exists $C > 0$ such that for all $k \in \mathbb{N}_0$

$$\left| \frac{1}{2}M^{[k]}(1) - 1 \right| \leq C \cdot 2^{-k} \quad (48)$$

At first we notice the following result:

Proposition 27 *Let $\lambda_0, \dots, \lambda_{n-1}$ be real numbers. Then there exists $k_0 \in \mathbb{N}_0$ such that the Daubechies type subdivision scheme reproduces stepwise functions in $E(\lambda_0, \dots, \lambda_{n-1})$ for all levels $k \geq k_0$.*

Proof. Let $z_j^{[k]} = \exp(-\lambda_j/2^{k+1})$. By construction $M^{[k]}(z)$ has a zero at $-z_j^{[k]}$ of multiplicity μ_j , the number of times λ_j occurs in $(\lambda_0, \dots, \lambda_{n-1})$, hence

$$\frac{d^s}{dx^s} M^{[k]}(-z_j^{[k]}) = 0 \text{ for } s = 0, \dots, \mu_j - 1 \text{ and } j = 1, \dots, n-1. \quad (49)$$

By (47) and the fact that $\{a^{[k]}\}$ reproduces stepwise functions in $E(\lambda_0, \dots, \lambda_{2n-1})$ we conclude that

$$\frac{1}{2} |M^{[k]}(z_j^{[k]})|^2 = a^{[k]}(z_j^{[k]}) = 2.$$

Since $z_j^{[k]}$ is real and $M^{[k]}(z)$ has real coefficients it follows that $M^{[k]}(z_j^{[k]})$ is real so $M^{[k]}(z_j^{[k]}) = 2$ or -2 . Since $z_j^{[k]}$ converges to 1 for $k \rightarrow \infty$ and $M^{[k]}(z_j^{[k]})$ converges to $M^{D_{2n}}(1) > 0$ there exists $k_0 \in \mathbb{N}_0$ such that $M^{[k]}(z_j^{[k]}) > 0$ for all $k \geq k_0$ and $j = 1, \dots, n-1$. Hence $M^{[k]}(z_j^{[k]}) = 2$ for all $k \geq k_0$. From (47) we infer that for real x

$$\frac{d^s}{dx^s} a^{[k]}(x) = \sum_{r=0}^s \binom{s}{r} \frac{d^r}{dx^r} M^{[k]}(x) \cdot \frac{d^{s-r}}{dx^{s-r}} M^{[k]}(x).$$

For $s = 1$ this means that $0 = M^{[k]}(z_j^{[k]}) \frac{d}{dx} M^{[k]}(z_j^{[k]}) + \frac{d}{dx} M^{[k]}(z_j^{[k]}) \cdot M^{[k]}(z_j^{[k]})$. Since $M^{[k]}$ has real coefficients and $z_j^{[k]}$ is real we conclude that $\frac{d}{dx} M^{[k]}(z_j^{[k]}) = 0$. Inductively we obtain that $\frac{d^s}{dx^s} M^{[k]}(z_j^{[k]}) = 0$ for $s = 1, \dots, \mu_j - 1$. ■

Proposition 28 *The product $\prod_{k=1}^{\infty} \frac{1}{2} M^{[k-1]}(e^{i\frac{\omega}{2^k}})$ converges.*

Proof. By construction $M^{[k]}(e^{i\omega})$ has real coefficients and $M^{[k]}(1) > 0$. Since $a^{[k]}(e^{i\omega}) \geq 0$ we infer from (28) that $|a^{[k]}(e^{i\omega})| \leq 2$, and therefore $|M_k(e^{i\omega})| \leq 2$. Moreover, it follows from (48) that

$$\sum_{k=1}^{\infty} \left| \frac{1}{2} M^{[k]}(1) - 1 \right| \leq C \sum_{k=1}^{\infty} 2^{-k}.$$

Proposition 6 finishes the proof. ■

In order to obtain asymptotic equivalence for the non-stationary Daubechies subdivision scheme we have to choose the filter $M^{[k]}(z)$ with more care:

Theorem 29 *Let $M^{D_{2n}}$ be the Daubechies filter of order n as defined above and let $\lambda_0, \dots, \lambda_{n-1}$ be real numbers. Then there exists a non-stationary Daubechies type subdivision scheme $\{M^{[k]}(z)\}$ which is exponentially asymptotically equivalent to $M^{D_{2n}}$ and reproduces $e^{\lambda_j x}$ stepwise for all $k \geq k_0$ and for $j = 0, \dots, n-1$.*

Proof. Note that $M^{D_{2n}}(z)$ has n zeros at -1 and $n-1$ other zeros, say $\alpha_1, \dots, \alpha_{n-1}$ which are of course zeros of the factor $b_{D_{2n}}(z)$ of the Deslauriers-Dubuc symbol $D_{2n}(z) = \left(\frac{1+z}{2}\right)^{2n} b_{D_{2n}}(z)$. Using (24) and Proposition 16 we see that $\alpha_1, \dots, \alpha_{n-1}$ are pairwise different and simple zeros of $D_{2n}(z)$. Recall that $a^{[k]}(z)$ is exponentially asymptotically equivalent to the symbols $D_{2n}(z)$. Then $z^{2n-1}a^{[k]}(z)$ are polynomials and $z^{2n-1}a^{[k]}(z)$ is obviously exponentially asymptotically equivalent to $z^{2n-1}D_{2n}(z)$. By Theorem 15 there exists a constant $C > 0$ and a zero $\alpha_j^{[k]}$ of $a^{[k]}(z)$ such that $|\alpha_j^{[k]} - \alpha_j| \leq C2^{-k}$ for all $k \in \mathbb{N}_0$ and $j = 1, \dots, n-1$. Take $k_0 \in \mathbb{N}_0$ large enough so that:

(i) for each $k \geq k_0$ the balls $|z - \alpha_j| \leq C2^{-k}$ have empty intersection with the unit circle,

(ii) they are pairwise disjoint for $j = 1, \dots, n-1$, and

(iii) they have empty intersection with the x -axis if α_j is a non-real zero.

Then for each $k \geq k_0$ there is for given α_j exactly one zero $\alpha_j^{[k]}$ with

$$|\alpha_j^{[k]} - \alpha_j| \leq C2^{-k} \text{ for } j = 1, \dots, n-1, \quad (50)$$

leading to a *unique choice* for $M^{[k]}(z)$ for $k \geq k_0$. Further the leading coefficient $c^{[k]}$ of the polynomial $z^{2n-1}M^{[k]}(z)$ is determined by the equation

$$M^{[k]}(1) = c^{[k]} \prod_{j=0}^{n-1} (1 + z_j^{[k]}) \cdot \prod_{j=1}^{n-1} (1 - \alpha_j^{[k]}). \quad (51)$$

Let c be the leading coefficient of $z^{2n-1}D_{2n}(z)$. By (48) and (50) and (51) it is easy to see that there exists $D > 0$ such that

$$|c^{[k]} - c| \leq D2^{-k}$$

for all $k \in \mathbb{N}_0$. By Theorem 14 the subdivision scheme defined by the symbols $z^{2n-1}M^{[k]}(z)$, $k \in \mathbb{N}_0$, is exponentially asymptotically equivalent to the scheme defined by the symbol $z^{2n-1}M^{D_{2n}}$. This implies that $M^{[k]}(z)$, $k \in \mathbb{N}_0$, is exponentially asymptotically equivalent to $M^{D_{2n}}$. ■

Remark 30 Assume that $M^{D_{2n}}$ is the Daubechies filter such that all zeros $\neq -1$ have absolute value bigger than 1. Then one can define $M^{[k]}(z)$ in the last theorem by the condition that all its non-trivial zeros have absolute value bigger than 1.

It thus follows, from the theory of asymptotically equivalent schemes in [16], that the scheme with symbols $\{M^{[k+m]}(z), k \in \mathbb{N}_0\}$ defines continuous basic limit functions $\{\varphi_m^{\Lambda_0}(\cdot)\}$. Proposition 7 shows that

$$\widehat{\varphi_m^{\Lambda_0}}(\omega) = \prod_{k=1}^{\infty} \frac{M^{[m+k-1]} \left(e^{i \frac{\omega}{2^k}} \right)}{2}. \quad (52)$$

In particular, we have

$$\left| \widehat{\varphi_m^{\Lambda_0}}(\omega) \right|^2 = \widehat{\Phi_m^{\Lambda_{2n}}}(\omega) \quad (53)$$

where $\Phi_m^{\Lambda_{2n}}$ is the basic limit function of $S_m^{\Lambda_{2n}}$.

The following observation is straightforward and the proof is included for convenience of the reader:

Proposition 31 *The functions $2^{m/2} \varphi_m^{\Lambda_0}(2^m \cdot -k), k \in \mathbb{Z}$, are orthonormal.*

Proof. Define $D_{k,l} := 2^m \int_{-\infty}^{\infty} \varphi_m^{\Lambda_0}(2^m t - k) \overline{\varphi_m^{\Lambda_0}(2^m t - l)} dt$. A simple transformation of variables and Plancherel's formula gives

$$D_{k,l} = \int_{-\infty}^{\infty} \varphi_m^{\Lambda_0}(y - k) \overline{\varphi_m^{\Lambda_0}(y - l)} dy = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{\varphi_m^{\Lambda_0}}(\omega) \overline{\widehat{\varphi_m^{\Lambda_0}}(\omega)} e^{-i(k-l)\omega} d\omega.$$

Using (53) we obtain that

$$D_{k,l} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_m^{\Lambda_{2n}}(\omega) e^{-i(k-l)\omega} d\omega = \Phi_m^{\Lambda_{2n}}(k - l) = \delta_{0,k-l}.$$

■

Definition 32 *For each $m \in \mathbb{N}_0$ we define the linear spaces V_m by*

$$V_m := \left\{ f \in L^2(\mathbb{R}) \mid f(t) = \sum_{j \in \mathbb{Z}} c_j \varphi_m^{\Lambda_0}(2^m t - j), \sum_{j \in \mathbb{Z}} |c_j|^2 < \infty \right\}. \quad (54)$$

We remark that we could also define V_m for integers $m \in \mathbb{Z}$ since the symbols $a^{[k+m]}(z)$ and the scaling function $\Phi_m^{\Lambda_{2n}}$ and $\varphi_m^{\Lambda_0}$ could be defined for all $m \in \mathbb{Z}$.

Proposition 33 *The spaces V_m are nested, i.e. that $V_m \subset V_{m+1}$ for all $m \in \mathbb{N}_0$.*

Proof. Using the product representation (52) we obtain

$$\widehat{\varphi_m^{\Lambda_0}}(\omega) = \prod_{k=1}^{\infty} \frac{M^{[m+k-1]} \left(e^{i \frac{\omega}{2^k}} \right)}{2} = \frac{M^{[m]} \left(e^{i \frac{\omega}{2}} \right)}{2} \widehat{\varphi_{m+1}^{\Lambda_0}} \left(\frac{\omega}{2} \right). \quad (55)$$

Let us write $M^{[m]}(z) = \sum_{j=-n+1}^n \mu_j^{[m]} z^j$ where $\mu_j^{[m]}$ are *real* numbers. Using elementary techniques in Fourier analysis it is easy to see that the equation (55) is equivalent to the refinement equation

$$\varphi_m^{\Lambda_0}(t) = \sum_{j=-n+1}^n \mu_j^{[m]} \varphi_{m+1}^{\Lambda_0}(2t + j). \quad (56)$$

Replacing t by $2^m t$ in (56) we obtain that $\varphi_m^{\Lambda_0}(2^m \cdot) \in V_{m+1}$. Similarly it follows that $\varphi_m^{\Lambda_0}(2^m t - j) \in V_{m+1}$ for each $j \in \mathbb{Z}$. ■

We note that the orthonormality of $\varphi_m^{\Lambda_0}(t - l)$ implies that

$$\delta_{0,l} = \int_{-\infty}^{\infty} \varphi_m^{\Lambda_0}(t) \overline{\varphi_m^{\Lambda_0}(t - l)} dt = \frac{1}{2} \sum_{j \in \mathbb{Z}} \mu_j^{[m]} \mu_{j+2l}^{[m]}. \quad (57)$$

Indeed, using (56) in the integral in (57) and then again the orthonormality relations one obtains

$$\begin{aligned} \delta_{0,l} &= \sum_{j,k \in \mathbb{Z}} \mu_j^{[m]} \overline{\mu_k^{[m]}} \int_{-\infty}^{\infty} \varphi_{m+1}^{\Lambda_0}(2t + j) \overline{\varphi_{m+1}^{\Lambda_0}(2t - 2l + k)} dt \\ &= \frac{1}{2} \sum_{j,k \in \mathbb{Z}} \mu_j^{[m]} \overline{\mu_k^{[m]}} \int_{-\infty}^{\infty} \varphi_{m+1}^{\Lambda_0}(y) \overline{\varphi_{m+1}^{\Lambda_0}(y - j - 2l + k)} dy. \end{aligned}$$

Proposition 34 *The union of the spaces V_m is dense in $L_2(\mathbb{R})$.*

Proof. Define $f_m(x) = \varphi_m^{\Lambda_0}(2^m x)$. Then $\widehat{f_m}(\omega) = \widehat{\varphi_m^{\Lambda_0}}(2^{-m}\omega)$. By Theorem 4.3 in [8] it suffices to show that the set Ω , defined as the union of the supports of $\widehat{f_m}$ over $m \in \mathbb{N}_0$, is equal to \mathbb{R} minus a set of Lebesgue measure 0. Since $\widehat{\Phi_m^{\Lambda_{2^n}}}(\omega) = |\widehat{\varphi_m^{\Lambda_0}}(\omega)|^2$ we have only to investigate the zeros of $\widehat{\Phi_m^{\Lambda_{2^n}}}(\omega)$. By (52) this is an infinite product of non-negative terms $a^{[m+k]} \left(e^{i\omega 2^{-k}} \right)$ which may be zero only if $e^{i\omega 2^{-k}} = -1$, see [30]. Hence, the zeros of $\widehat{\Phi_m^{\Lambda_{2^n}}}$ are of the form $\omega = 2^k(\pi + 2n\pi)$ for some $n \in \mathbb{Z}$. Hence the support of $\widehat{\Phi_m^{\Lambda_{2^n}}}$ is equal to the real line. This ends the proof. ■

Theorem 35 *Let $M^{D_{2^n}}(z)$ be a Daubechies filter of order n and assume that $M^{[k]}(z)$ is as in Theorem 29. Let $\alpha \in [0, 1)$ and $\ell \in \mathbb{N}_0$, and assume that the scaling function $\varphi^{D_{2^n}}$ of Daubechies defined by the symbol $M^{D_{2^n}}(z)$, satisfies for some $\varepsilon > 0$ and $C > 0$ the inequality*

$$|\widehat{\varphi^{D_{2^n}}}(\omega)| \leq C(|\omega| + 1)^{-\ell-1-\alpha-\varepsilon}$$

for all $\omega \in \mathbb{R}$. Then the scaling function $\varphi_m^{\Lambda_0}$ associated to the subdivision scheme $M^{[k+m]}(z)$, $k \in \mathbb{N}_0$, have ℓ -th derivative in $Lip(\alpha)$.

Proof. Let us define $z_j^{[k]} = \exp(-2^{-(k+1)}\lambda_j)$. Then the symbol $M^{[k]}(z)$ can be written as

$$M^{[k]}(z) = \left(\prod_{j=1}^n \frac{z + z_j^{[k]}}{2} \right) B^{[k]}(z).$$

Similarly we can write for the Daubechies filter

$$M^{D_{2^n}}(z) = \left(\prod_{j=1}^n \frac{z + 1}{2} \right) B^{D_{2^n}}(z).$$

Since $M^{[k]}(z)$ is exponentially asymptotically equivalent to the Daubechies filter $M^{D_{2n}}$ and $B^{[k]}(z)$ has only simple zeros it follows from Theorems 15 and 14 that $B^{[k]}$ is exponentially asymptotically equivalent to $B^{D_{2N}}(z)$, so there exists a constant $C > 0$ such that

$$|B^{[k]}(e^{i\omega}) - B^{D_{2N}}(z)(e^{i\omega})| \leq C \cdot 2^{-k}$$

Now one can proceed as in Theorem 22. ■

Next we turn to the construction of Daubechies type wavelets. The procedure will follow the classical pattern in MRA. For convenience of the reader we shall briefly sketch the construction: Recall that $M^{[m]}(z) = \sum_{j=-n+1}^n \mu_j^{[m]} z^j$ and we write the refinement equation in (56) in the form used in MRA, namely

$$\varphi_m^{\Lambda_0}(t) = \sum_{j \in \mathbb{Z}} \mu_{-j}^{[m]} \varphi_{m+1}^{\Lambda_0}(2t - j).$$

The Daubechies type wavelets $\psi_m^{\Lambda_0}$ are now defined in the classical way, namely by

$$\psi_m^{\Lambda_0}(t) = \sum_{j \in \mathbb{Z}} \nu_j^{[m]} \varphi_{m+1}^{\Lambda_0}(2t - j), \quad (58)$$

where the coefficients $\{\nu_k^{[m]}\}$ are related to those in (56) by

$$\nu_j^{[m]} = (-1)^{j+1} \mu_{-1+j}^{[m]}. \quad (59)$$

Then $\psi_m^{\Lambda_0}$ has compact support since $\varphi_{m+1}^{\Lambda_0}$ has compact support. It is a routine exercise to see that the system of functions $\{2^{m/2} \psi_m^{\Lambda_0}(2^m t - r) : r \in \mathbb{Z}\}$ is orthonormal: define

$$D_{r,s} := 2^m \int_{-\infty}^{\infty} \psi_m^{\Lambda_0}(2^m t - r) \overline{\psi_m^{\Lambda_0}(2^m t - s)} dt.$$

The transformation $y = 2^m t - r$ gives $D_{r,s} = \int_{-\infty}^{\infty} \psi_m^{\Lambda_0}(y) \overline{\psi_m^{\Lambda_0}(y + r - s)} dy$ and therefore

$$\begin{aligned} D_{r,s} &= \sum_{k,l \in \mathbb{Z}} \nu_k^{[m]} \nu_l^{[m]} \int_{-\infty}^{\infty} \varphi_{m+1}^{\Lambda_0}(2y - k) \overline{\varphi_{m+1}^{\Lambda_0}(2y + 2r - 2s - l)} dy \\ &= \frac{1}{2} \sum_{k,l \in \mathbb{Z}} \nu_k^{[m]} \nu_l^{[m]} \int_{-\infty}^{\infty} \varphi_{m+1}^{\Lambda_0}(x) \overline{\varphi_{m+1}^{\Lambda_0}(x + 2r - 2s + k - l)} dx \\ &= \frac{1}{2} \sum_{k \in \mathbb{Z}} \nu_k^{[m]} \nu_{k+2(r-s)}^{[m]} = \frac{1}{2} \sum_{k \in \mathbb{Z}} \mu_{-1+k}^{[m]} \mu_{-1+k+2s-2r}^{[m]} = \delta_{0,r-s} \end{aligned}$$

where we have used equation (57).

As explained in Section 4 the wavelet space W_m is defined as the orthogonal complement W_m of V_m in V_{m+1} . Next one shows that the wavelet space W_m is equal to

$$\widetilde{W}_m := \left\{ f \in L^2(\mathbb{R}) \mid f(t) = \sum_{j \in \mathbb{Z}} c_j 2^{m/2} \psi_m^{\Lambda_0}(2^m t - j), \quad \sum_{j \in \mathbb{Z}} |c_j|^2 < \infty \right\}. \quad (60)$$

We show at first that \widetilde{W}_m is orthogonal to $V_m \subset V_{m+1}$ implying that $\widetilde{W}_m \subset W_m$. So we look at

$$C_{r,s} := 2^m \int_{-\infty}^{\infty} \varphi_m^{\Lambda_0}(2^m t - r) \overline{\psi_m^{\Lambda_0}(2^m t - s)} dt.$$

Again $C_{r,s} = \int_{-\infty}^{\infty} \varphi_m^{\Lambda_0}(y) \overline{\psi_m^{\Lambda_0}(y + r - s)} dy$ and (56) and (58) yield

$$\begin{aligned} C_{r,s} &= \sum_{j,l \in \mathbb{Z}} \mu_{-j}^{[m]} \nu_l^{[m]} \int_{-\infty}^{\infty} \varphi_{m+1}^{\Lambda_0}(2y - j) \overline{\varphi_{m+1}^{\Lambda_0}(2y + 2r - 2s - l)} dy \\ &= \frac{1}{2} \sum_{j,l \in \mathbb{Z}} \mu_{-j}^{[m]} \nu_l^{[m]} \int_{-\infty}^{\infty} \varphi_{m+1}^{\Lambda_0}(x) \overline{\varphi_{m+1}^{\Lambda_0}(x + j + 2r - 2s - l)} dx \\ &= \frac{1}{2} \sum_{j \in \mathbb{Z}} \mu_{-j}^{[m]} \nu_{j+2r-2s}^{[m]} = -\frac{1}{2} \sum_{j \in \mathbb{Z}} \mu_{-j}^{[m]} (-1)^j \mu_{-1+j+2r-2s}^{[m]}. \end{aligned}$$

A simple well known argument shows that the last sum is always zero, see e.g. [27, p. 123]. The proof that $\widetilde{W}_m \oplus V_m$ is equal to V_{m+1} (implying that $W_m = \widetilde{W}_m$) follows standard arguments in MRA and is omitted.

Corollary 36 *The smoothness of the Daubechies type wavelets $\psi_m^{\Lambda_0}$ is at least as that of the classical Daubechies wavelet ψ*

Proof. This follows immediately from formula (58) and Proposition 35. ■

Finally we mention that the concept of reproduction is defined in MRA in the following way, see e.g. [36]:

Definition 37 *A non-stationary multiresolutional analysis $(V_m)_{m \in \mathbb{Z}}$ with compactly supported scaling functions φ_m reproduces a function $f : \mathbb{R} \rightarrow \mathbb{C}$ if for each $m \in \mathbb{Z}$ there exist complex coefficients c_m such that*

$$f(x) = \sum_{l \in \mathbb{Z}} c_m \varphi_m(2^m x - l). \quad (61)$$

Note that the sum in (61) is well defined since φ_m is compactly supported. It is proved in [36] that (49) implies that the MRA $(V_m)_{m \in \mathbb{Z}}$ of the Daubechies subdivision scheme reproduces the space $E(\lambda_0, \dots, \lambda_{n-1})$ and the wavelets ψ_m have vanishing moments in the sense that

$$\int \psi_m^{\Lambda_0}(t) e^{\lambda_j t} dt = 0 \quad \text{for all } j = 0, \dots, n-1.$$

As we mentioned in the introduction an attempt to construct (non-stationary) Daubechies type wavelets based on the subdivision scheme for exponential polynomials in analogy to the case of Deslauriers and Dubuc, was carried out in [36]. The authors showed that many results and techniques of classical multiresolutional analysis, briefly MRA, carry

over to non-stationary MRA (except that all filters and scaling functions are dependent on the scale of the multiresolution) and they introduced non-stationary Daubechies-type wavelets reproducing a family of exponentials $\{e^{\lambda_j t}\}_{j=1}^n$ under the assumption that the filters have sufficiently long support. This strong assumption has been made in order to guarantee that the symbols of the nonstationary scheme are non-negative. However, as we mentioned in the introduction, a main result proven in [30] shows that the symbols are non-negative under the simple assumption that the exponents λ_j for $j = 1, \dots, n$ are real. As a consequence, the question about the order of regularity of the new Daubechies type wavelets, is not addressed in [36].

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